

# Delicate $f(R)$ gravity models with disappearing cosmological constant and observational constraints on the model parameters

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We study the  $f(R)$  theory of gravity using metric approach. In particular we investigate the recently proposed model by Hu-Sawicki, Appleby – Battye and Starobinsky. In this model, the cosmological constant is zero in flat space time. The model passes both the Solar system and the laboratory tests. But the model parameters need to be fine tuned to avoid the finite time singularity recently pointed in the literature. We check the concordance of this model with the  $H(z)$  and baryon acoustic oscillation data. We find that the model resembles the  $\Lambda$ CDM at high redshift. However, for some parameter values there are variations in the expansion history of the universe at low redshift.

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## I. INTRODUCTION

It is remarkable that different data sets of complementary nature such as supernovae, baryon oscillations, galaxy clustering, microwave back ground and weak lensing all taken together strongly support the late time acceleration of universe. In the standard lore, one assumes that the history of universe is described by the general relativity (GR). The late time acceleration can easily be captured in this frame work by introducing a scalar field with large negative pressure known as *dark energy* [1]. In view of the fine tuning problem, the scalar field models, specially those with tracker like solutions, are more attractive compared to the models based on cosmological constant. At present, observations are not in a position to reject or to establish the dark energy metamorphosis. A host of scalar field models have been investigated in the literature. The scalar field models can fit the data but lack the predictive power. It then becomes important to seek the support of these models from a fundamental theory of high energy physics.

One can question the standard lore on fundamental grounds. We know that gravity is modified at small distance scales; it is quite possible that it is modified at large scales too where it has never been confronted with observations directly. It is therefore perfectly legitimate to investigate the possibility of late time acceleration due to modification of Einstein-Hilbert action. It is tempting to study the string curvature corrections to Einstein gravity amongst which the Gauss-Bonnet correction enjoys special status. A large number of papers are devoted to the cosmological implications of string curvature corrected gravity[2, 3, 4, 5, 6, 7, 8, 9, 10, 11]. These models suffer from several problems. Most of these models do not include tracker like solution and those which do are heavily constrained by the thermal history of universe. For instance, the Gauss-Bonnet theory with dynamical dilaton might cause transition from matter scaling regime to late time acceleration allowing to alleviate the fine tuning and coincidence problems. However, it is difficult to reconcile this model with nucleosynthesis[4, 5]. Another possibility of large scale modification is provided by non-locally corrected gravity which typically involves inverse of d'Alembertian of Ricci scalar. The non-local construct might mimic dark energy; the model poses technical difficulties and there has been a little progress in this direction[12]. The large scale modification may also arise in extra dimensional theories like DGP model which contains self accelerating brane. Apart from the theoretical problems, this model is heavily constrained by observation.

On purely phenomenological grounds, one could seek a modification of Einstein gravity by replacing the Ricci scalar by  $f(R)$ . The  $f(R)$  gravity models have been extensively investigated in past five years[13, 14, 15, 16]. The  $f(R)$  gravity theories giving rise to cosmological constant in low curvature regime are plagued with instabilities and on observational grounds they are not distinguished from cosmological constant. The recently introduced models of  $f(R)$  gravity by Hu-Sawicki and Starobinsky (referred as HSS models hereafter) with disappearing cosmological constant[17, 18] have given rise to new hopes for a viable cosmological model within the framework of modified gravity ( $f(R)$  gravity model with similar properties is proposed in Ref.[19]). These models contain Minkowski space time as a solution in the low curvature regime which is an unstable solution. In high curvature regime these models reduce to cosmological constant. Both the first and the second derivatives of  $f(R)$  with respect to  $R$  are positive. The positivity of the first derivative ensures that the scalar degree of freedom, a characteristic of any  $f(R)$  theory, is not tachyonic where as the

positivity of second derivative tells us that graviton is not ghost thereby guaranteeing the stability. In Starobinsky parametrization[18],  $f(R)$  is given by,  $f(R) = R + \Lambda \left[ (1 + R^2/R_0^2)^{-n} - 1 \right]$ . The HSS models can evade solar physics constraints provided that the model parameters are chosen properly. An important observation has recently been made by Appleby – Battye and Forolov[21, 22] (see also [23]). The minimum of scalaron potential which corresponds to dark energy can be very near to  $\phi = 0$  or equivalently  $R = \infty$ . As pointed out in Ref.[24], the minimum should be near the origin for solar constraints to be evaded. Hence, it becomes most likely that we hit the singularity if the parameters are not fine tuned.

In order to check whether the  $f(R)$  gravity theory is cosmological viable or not, it is necessary that this theory must be compatible with the observations. In this work, we study the Starobinsky model using the data from the recent observations which include  $H(z)$ , Hubble parameter at various red-shifts and the Baryon Acoustic Oscillation (BAO) peak from Sloan Digital Sky Survey (SDSS).

This paper is organised as follows. In Section II, we describe the general properties of the model highlighting the fine tuning problem. The Friedmann equation and the special cases of trace equation is studied in Section III. The cosmological constraints from the recent observations are described in Section IV. Finally, Section V contains the results and discussions.

## II. NATURALNESS OF THE MODEL

In this section we shall revisit  $f(R)$  models with disappearing cosmological constant. These models have potential capability of being distinguished from  $\Lambda$ CDM and could lead to a viable cosmological model. However, even at the background level this class of models are fine tuned. In what follows we shall explicitly bring out these features specializing to Starobinsky parametrization.

The action of  $f(R)$  gravity is given by[14],

$$S = \int \left[ \frac{f(R)}{16\pi G} + \mathcal{L}_m \right] \sqrt{-g} \, d^4x, \quad (1)$$

which leads to the following equation of motion

$$f' R_{\mu\nu} - \nabla_{\mu\nu} f' + \left( \square f' - \frac{1}{2} f' \right) g_{\mu\nu} = 8\pi G T_{\mu\nu}. \quad (2)$$

Here prime (') denotes the derivatives with respect to  $R$ . The  $f(R)$  gravity theories apart from a spin-2 object necessarily contain a scalar degree of freedom. Taking trace of Eq.(2) gives the evolution equation for the scalar degree of freedom,

$$\square f' = \frac{1}{3} (2f - f' R) + \frac{8\pi G}{3} T. \quad (3)$$

It would be convenient to define scalar function  $\phi$  as

$$\phi \equiv f' - 1, \quad (4)$$

which is expressed through Ricci scalar once  $f(R)$  is specified.

We can write the trace equation (equation (3)) in the term of  $V$  and  $T$  as

$$\square \phi = \frac{dV}{d\phi} + \frac{8\pi G}{3} T. \quad (5)$$

The potential can be evaluated using the following relation

$$\frac{dV}{dR} = \frac{dV}{d\phi} \frac{d\phi}{dR} = \frac{1}{3} (2f - f' R) f''. \quad (6)$$

Recently Hu, Sawicki and Starobinsky proposed a functional form of  $f(R)$  with the desirable properties of a viable model [17, 18]. In this paper we shall consider the model in Starobinsky parametrization:

$$f(R) = R + \lambda R_0 \left[ \left( 1 + \frac{R^2}{R_0^2} \right)^{-n} - 1 \right]. \quad (7)$$

Here  $n$  and  $\lambda$  are greater than zero. And  $R_0$  is of the order of presently observed cosmological constant,  $\Lambda = 8\pi G\rho_{vac}$ . The properties of this model can be summarized as follows:

1. In the absence of matter  $R_\nu^\mu$  is always a solution,  $\lim_{R \rightarrow 0} f(R) = 0$ . However, as  $f'' < 0$ , flat space time is unstable.
2. For  $|R| \gg R_0$ ,  $f(R) = R - 2\Lambda(\infty)$ . The high-curvature value of the effective cosmological constant is  $\Lambda(\infty) = \lambda R_0/2$ .
3. The stability conditions for the adopted model are

$$f'(R) > 0, \quad f''(R) > 0. \quad (8)$$

In the Starobinsky model the scalar field  $\phi$  is given by

$$\phi(R) = -\frac{2n\lambda R}{R_0(1 + \frac{R^2}{R_0^2})^{n+1}}. \quad (9)$$

We can compute  $V(R)$  for a given value of  $n$ . In case of  $n = 1$ , we have

$$\begin{aligned} \frac{V}{R_0} = & \frac{1}{24(1+y^2)^4} \{ (-8 - 40y^2 - 56y^4 - 24y^6) \lambda + (3y + 11y^3 + 21y^5 - 3y^7) \lambda^2 \} \\ & - \frac{\lambda^2}{8} \tan^{-1} y, \end{aligned} \quad (10)$$

where  $y = R/R_0$ . In case and of  $n = 2$  we have,

$$\begin{aligned} \frac{V}{R_0} = & \frac{1}{480(1+y^2)^6} \{ (-160 - 1280y^2 - 2880y^4 - 2560y^6 - 800y^8) \lambda \\ & + (105y + 595y^3 + 2154y^5 + 106y^7 + 595y^9 + 105y^{11}) \lambda^2 \} \\ & - \frac{35\lambda^2}{160} \tan^{-1} y. \end{aligned} \quad (11)$$

In the FRW background, the trace equation (equation (3)) can be rewritten in the convenient form

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = \frac{8\pi G}{3}\rho. \quad (12)$$

The time-time component of the equation of motion (2) gives the Hubble equation

$$H^2 + \frac{d(\ln f')}{dt} H + \frac{1}{6} \frac{f - f'R}{f'} = \frac{8\pi G}{3f'} \rho. \quad (13)$$

We recover Einstein gravity in the limit  $f' = 1$ . The simple picture of dynamics which appears here is the following: above infrared modification scale ( $R_0$ ), the expansion rate is set by the matter density and once the local curvature falls below  $R_0$  the expansion rate gets effect of gravity modification.

For pressure less dust, the effective potential has an extremum at

$$2f - Rf' = 8\pi G\rho. \quad (14)$$

For a viable late time cosmology, the field should be evolving near the minimum of the effective potential. The finite time singularity inherent in the class of models under consideration severely constraints dynamics of the field.

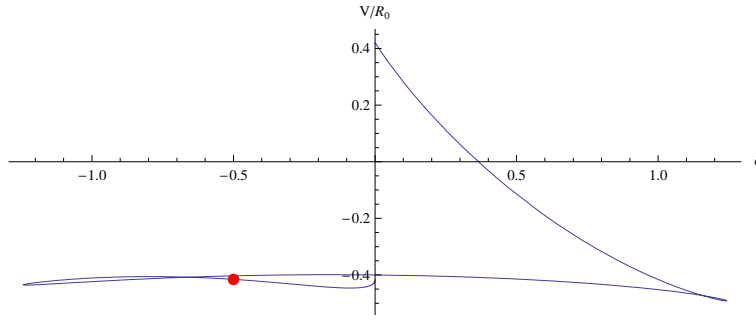


FIG. 1: Plot of the scalaron potential  $V$  versus  $\phi$  for  $n = 2$  and  $\lambda = 1.2$ . The red spot marks the initial condition for evolution of  $\phi$ .

### The finite time singularity and fine tuning of parameters

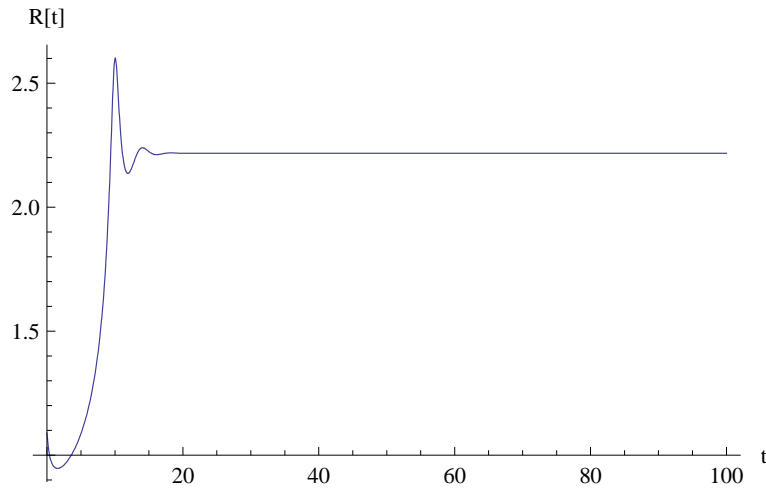


FIG. 2: The evolution of  $R(t)$  in the case  $n = 2$  and  $\lambda = 1.2$ . The initial condition is as marked by the red spot in figure 1.

The effective potential has minimum which depends upon  $n$  and  $\lambda$ . For generic values of the parameters, the minimum of the potential is close to  $\phi = 0$ , corresponding to infinitely large curvature. Thus, while the field is evolving towards minimum, it can easily oscillate to a singular point. However, depending upon the values of parameters, we can choose a finite range of initial conditions for which scalar field  $\phi$  can evolve to the minimum of the potential without hitting the singularity. For instance, let us begin with the initial conditions as shown in figure 1 given by  $\phi_{int} = -0.5$  ( $R = 1.089$ ). We choose  $H = 1$  then  $\dot{H}$  becomes  $-1.818$  for a given value of  $R$  (see eq. (13)). The minimum of the potential for  $n = 2$  and  $\lambda = 1.2$  corresponds to  $R_{min} = 2.218$  or  $\phi_{min} = -0.051$  which is indeed close to singularity. The numerical results are shown in figures 1 and 2.

### The problem of large $n, \lambda$

We find that the range of initial conditions allowed for the evolution of  $\phi$  to the minimum without hitting singularity shrinks as the numerical values of parameters  $n$  and  $\lambda$  increase. This is related to the fact that for larger values of  $n$  and  $\lambda$ , the minimum fast moves towards  $\phi = 0$ , see figure 3.

For example, in the case of  $n = 1, \lambda = 4$ , we find that  $\phi_{min} = -0.016$  which is close to zero. The initial value of  $\phi$  which is quite near to minimum also gives the divergence of  $H(t)$  and  $R(t)$  at the finite time (see Fig. 4).

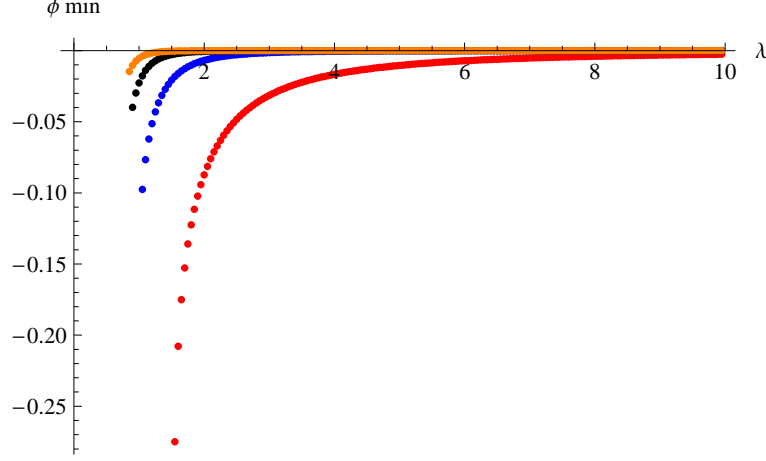


FIG. 3: Plot of  $\phi_{min}$  versus  $\lambda$  for different values of  $n$ . With increase in  $n$ ,  $\phi_{min}$  moves towards zero (corresponding to infinitely large value of  $R$ ) for smaller values of  $\lambda$ . The curves from bottom to top correspond to  $n = 1, 2, 3, 4$  respectively.

On the other hand, one can see from the figure 3 that  $\phi_{min}$  also shifts to zero when  $n$  is increased. For  $n = 4$  and  $\lambda = 1.1$  we have  $\phi_{min} = -0.003$ . The range of initial conditions for which the scalar field can evolve to the minimum is very small.

### Adding $\alpha R^2$ term and curing the singularity

We know that in case of large curvature, the quantum effects become important leading to higher curvature corrections. Keeping this in mind, let us consider the modification of Starobinsky's model,

$$f(R) = R + \frac{\alpha}{R_0} R^2 + R_0 \lambda \left[ -1 + \frac{1}{(1 + \frac{R^2}{R_0^2})^n} \right], \quad (15)$$

then  $\phi$  becomes

$$\phi(R) = \frac{R}{R_0} \left[ 2\alpha - \frac{2n\lambda}{(1 + \frac{R^2}{R_0^2})^{n+1}} \right]. \quad (16)$$

When  $|R|$  is large the first term which comes from  $\alpha R^2$  dominates. In this case, the curvature singularity  $R = \pm\infty$  corresponds to  $\phi = \pm\infty$ . Hence, in this modification, the minimum of the effective potential is separated from the curvature singularity by the infinite distance in the  $\phi, V(\phi)$  plane.

For  $n = 2$ ,  $\phi$  and  $V(\phi)$  are given by

$$\phi(y) = 2\alpha y - \frac{4\lambda y}{(1 + y^2)^3}, \quad (17)$$

$$\begin{aligned} \frac{V}{R_0} = & -\frac{1}{480(1 + y^2)^6} \{ \lambda^2 y (-105 - 595y^2 - 2154y^4 + 106y^6 + 595y^8 + 105y^{10}) \} \\ & - \frac{1}{3(1 + y^2)^3} \{ 1 + 5y^2 + \alpha (3 + 8y^2 + 9y^4 + 4y^6) \} + \frac{1}{3}\alpha y^2 \\ & + \frac{1}{32} (32\alpha - 7\lambda) \tan^{-1}(y). \end{aligned} \quad (18)$$

For  $n = 2, \lambda = 2$  and  $\alpha = 0.5$ , we have a large range of the initial condition for which the scalar field evolves to the minimum of the potential. Though the introduction of  $R^2$  term formally allows to avoid the singularity but can not

alleviate the fine tuning problem as the minimum should be brought near to the origin to respect the solar constraints. Last but not the least one could go beyond the approximation (see equation (14)) by iterating the trace equation and computing the corrections to  $R$  given by equation (14). As pointed by Starobinsky[18], such a correction might become large in the past. This may spoil the thermal history and thus needs to be fine tuned. The aforesaid discussion makes it clear that HSS models are indeed fine tuned and hence very delicate.

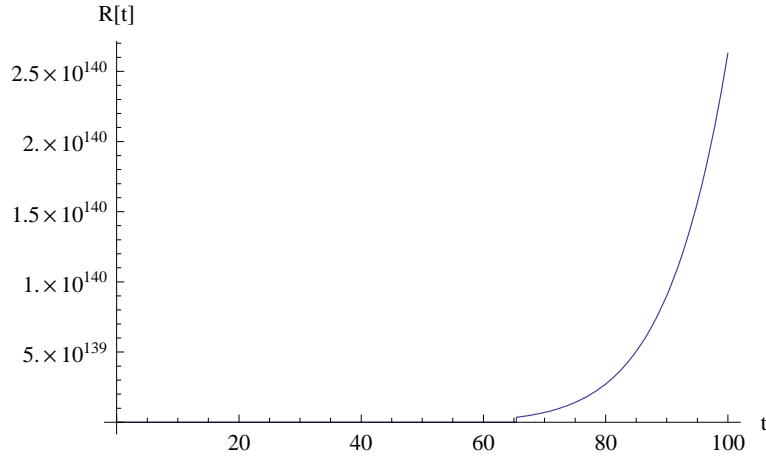


FIG. 4: The evolution of  $R(t)$  for  $n = 1$  and  $\lambda = 4$ .

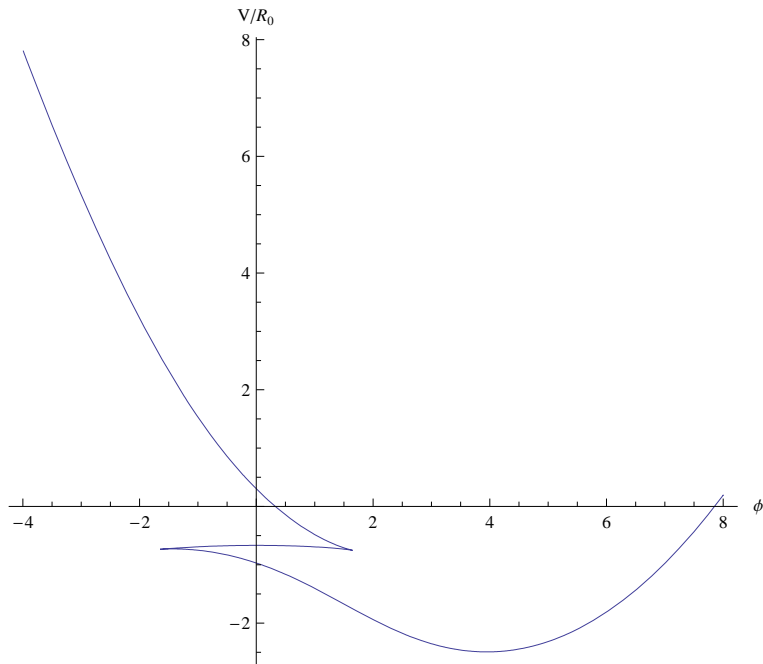


FIG. 5: Plot of the effective potential for  $n = 2$ ,  $\lambda = 2$  and  $\alpha = 1/2$  in presence of  $R^2$  correction. The minimum of the effective potential in this case is located at  $\phi_{min} = 3.952$  ( $R_{min} = 3.958$ ).

### III. PARAMETRIZATION AND ANALYSIS

The Friedmann equation can be casted in the form,

$$H^2 = \frac{6\kappa\rho + Rf' - f}{6(f' + R_{,N}f'')}, \quad (19)$$

where  $N = \ln a$ . To parameterise the system in a more convenient form we define:

$$\kappa\rho = \Omega_m H_0^2, \quad R_0 = 3\Omega_\Lambda H_0^2, \quad x = \frac{R_0}{R}. \quad (20)$$

Since  $|R| \gg R_0$  we have  $x \ll 1$ . The  $f, f'$  and  $f''$  for the Starobinsky model are of the form

$$\begin{aligned} \frac{f}{H_0^2} &= 3\Omega_\Lambda \left[ \frac{1}{x} - \lambda + \lambda \left( 1 + \frac{1}{x^2} \right)^{-n} \right], \\ f' &= 1 - \frac{2n\lambda}{x} \left( 1 + \frac{1}{x^2} \right)^{-(1+n)}, \\ H_0^2 f'' &= \frac{2n\lambda}{3\Omega_\Lambda} \left[ -1 + \frac{(2n+1)}{x^2} \right] \left( 1 + \frac{1}{x^2} \right)^{-(2+n)}. \end{aligned} \quad (21)$$

The trace equation can be re-written as

$$\frac{1}{x} + 2\lambda \left[ \left( 1 + \frac{1}{x^2} \right)^{-n} + \frac{n}{x^2} \left( 1 + \frac{1}{x^2} \right)^{-n-1} - 1 \right] = \frac{\Omega_m}{\Omega_\Lambda}. \quad (22)$$

Here  $\Omega_m = \Omega_{m0}(1+z)^3$  is the matter density of the universe at a given redshift.  $\Omega_{m0}$ , is present fractional matter density and  $\Omega_\Lambda$ , is the present fractional density of the vacuum.

The Friedmann equation (equation (19)) can now be written as

$$\frac{H^2}{H_0^2} = \frac{f/H_0^2 + 3\Omega_m}{6f' - \frac{54x\Omega_m H_0^2 f''}{(xf' - 3\Omega_\Lambda H_0^2 f'')}}. \quad (23)$$

Where  $f(R)$  and its derivatives are as given above. Note that we can recover the usual Friedmann equation for  $f(R) = R$ , the case for which our action reduces to Einstein Hilbert action.

In this paper we test the viability of  $f(R)$  cosmology. The Friedmann equation evaluated at  $z = 0$  imposes a constraint on the free parameters  $n$  and  $\lambda$ .

To check the compatibility of Starobinsky model with observations we need to first obtain the Hubble parameter  $H$ , as a function of redshift  $z$ . The Hubble equation in its present form depends on the curvature  $R$ , through  $f$  and its derivatives. Therefore, we need to solve the trace equation (equation (22)), relating curvature with redshift. In order to study the behavior of trace equation, which does not admit a simple solution for general  $n$ , we first analyse its quantitative behavior for large values of  $z$ , corresponding to  $x \ll 1$ . Therefore, the trace equation can be simplified considerably and in the leading order we recover

$$x = \frac{\Omega_\Lambda}{\Omega_{m0}(1+z)^3 + 2\lambda\Omega_\Lambda}. \quad (24)$$

Similarly, the Friedmann equation in the leading order gives

$$\frac{H^2}{H_0^2} = \Omega_{m0}(1+z)^3 + \frac{\lambda}{2}\Omega_\Lambda. \quad (25)$$

Thus at large redshift we recover the standard  $\Lambda$ CDM cosmology for  $\lambda = 2$ , independently of  $n$  (see figure 9).

There is no general solution for the trace equation and therefore we analyse this model for fixed values of  $n$ . It was shown by Capozziello and Tsujikawa [24] that we need  $n > 0.9$  to satisfy solar system tests. In this work we consider model with  $n = 1$  and  $2$ .

### A. $n=1$

The trace equation in this case simplifies to a quintic equation

$$F(x) \equiv x^5 - \frac{\Omega_\Lambda}{\Omega_m} x^4 + 2x^3 - 2\frac{\Omega_\Lambda}{\Omega_m} x^2 + x \left( 1 + 2\lambda \frac{\Omega_\Lambda}{\Omega_m} x^4 \right) - \frac{\Omega_\Lambda}{\Omega_m} = 0, \quad (26)$$

where  $\Omega_m = \Omega_{m0}(1+z)^3$ . This equation cannot be solved analytically for general  $x$  in terms of other parameters. Since  $x \ll 1$ , we approximate this equation with a quartic equation. This approximation is found to be sufficiently accurate for  $x \ll 1$  (see Fig. 6).

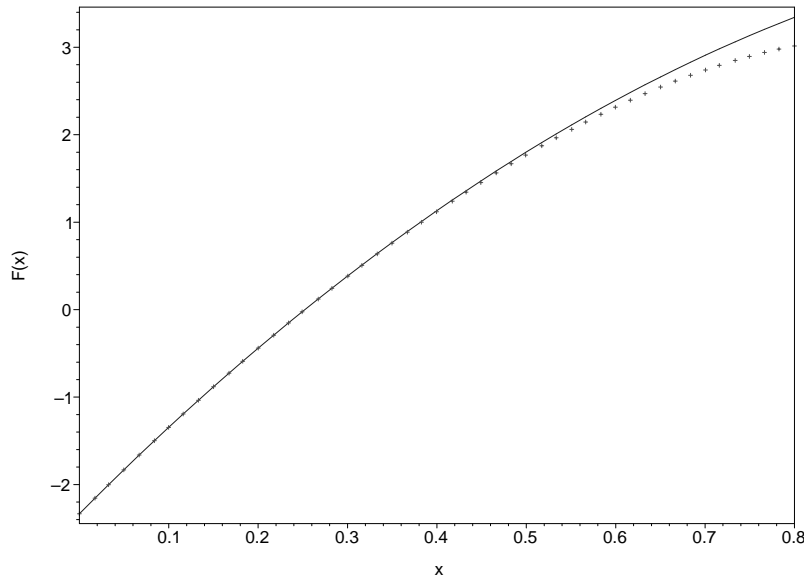


FIG. 6: Plot of  $F(x)$  at  $z=0, \lambda=2$  (indicated by solid line) vs truncated expression of  $F(x)$  at  $z=0, \lambda=2$  (indicated by dotted line).

We need to solve for  $x$  ( $x = R_0/R$ ) using the trace equation. We have two quartic equations  $F(x, z, \lambda) = 0$  and  $G(x, z, \lambda) = 0$  for  $n=1$  and 2 case, respectively. We shall solve the quartic equation with  $\Omega_{m0} = 0.3$  and  $\Omega_\Lambda = 0.7$ . Let us first consider the case of  $n=1$  and look for the roots of  $F=0$  at  $z=0$ . We numerically analyse the Eq. 26 and find that for generic values of  $\lambda$  ( $0 < \lambda < 10$ ), two of the four roots of quartic equation are always imaginary. The Friedman equation puts a constraints on the values of  $\lambda$  such that  $H(z) \rightarrow H_0$  for  $z \rightarrow 0$ , see Eq. 23. This condition fixes the value of  $\lambda$  to 2 for one of the roots, the other root corresponds to  $\lambda \simeq 2.44$ . We should now check the viability of the roots by invoking the non-zero values of the redshift. As  $z$  increases, i.e., we move to past,  $x = R_0/R$  should decrease as  $R$  should increase during a viable cosmic evolution which happens in case of  $\lambda=2$ . However, for the root corresponding to  $\lambda \simeq 2.44$ ,  $x$  increases in the past and  $x$  quickly goes beyond its normal range  $0 < x < 1$  and we need not consider this root at all. Therefore we have only one root obtained at  $\lambda=2$  which produces the correct final asymptotic state as a de Sitter model when  $z$  goes to  $-1$  (see figure 7). In this figure we plot the effective equation of state with redshift:

$$w_{eff} = -1 + \frac{2(1+z)}{3H} \frac{dH}{dz}. \quad (27)$$

Once  $\lambda$  is fixed there is no other free parameter in the theory. The Hubble parameter can now be plotted for the best fit value of  $\lambda$ .



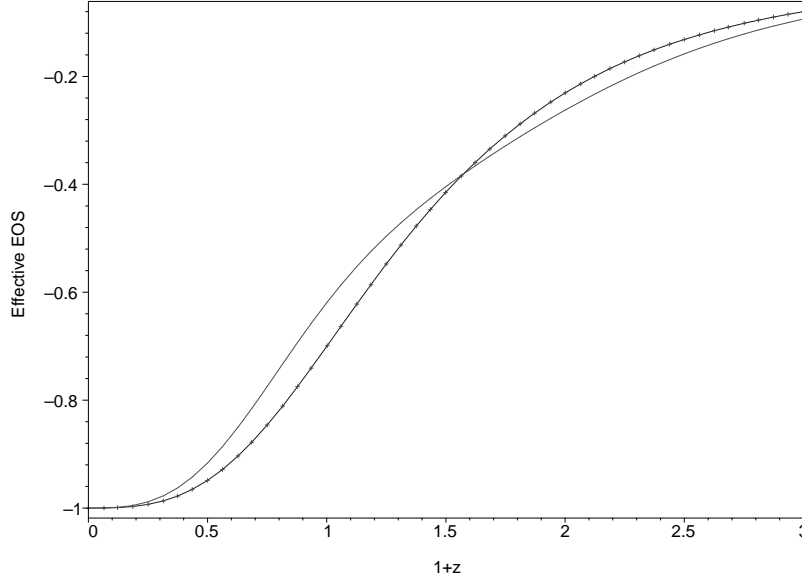


FIG. 7: Effective Equation of State (EOS) vs.  $1+z$ . Solid line corresponds to  $n=1$  and the knotted line corresponds to  $n=2$ .

### B. $n=2$

The trace equation in this case is a seventh degree polynomial in  $x$

$$G(x) \equiv x^7 - \frac{\Omega_\Lambda}{\Omega_m} x^6 + 3x^5 - 3\frac{\Omega_\Lambda}{\Omega_m} x^4 + 3\left(1 + 2\lambda\frac{\Omega_\Lambda}{\Omega_m}\right)x^3 - 3\frac{\Omega_\Lambda}{\Omega_m} x^2 + \left(1 + 2\lambda\frac{\Omega_\Lambda}{\Omega_m}\right)x - \frac{\Omega_\Lambda}{\Omega_m} = 0. \quad (28)$$

Again we can approximate this equation by a quartic equation in  $x$  using the fact that  $x \ll 1$  (see Fig.8). In order to calculate the roots of this equation, we followed the same procedure as described for  $n = 1$  case. Here also  $\lambda = 2$  reproduces the correct late time behavior (see figure 7).

## IV. OBSERVATIONAL CONSTRAINTS

In this paper, we check the compatibility of the model with the  $H(z)$  data and the measurements of baryon acoustic oscillation peak. We find the values of  $\chi^2$  to see how well the  $f(R)$  model with  $n = 1$  and  $n = 2$  accommodate the observations. Here, for a given model, we calculate  $\chi^2$  using  $H(z)$  data, BAO measurement and for the joint data ( $H(z) + \text{BAO}$ ).

### Constraints from $H(z)$ data

Simon, Verde and Jimenez (2005) employed differential ages of passively evolving galaxies to get the Hubble parameter as a function of redshift,  $H(z)$  [25]. The nine data points of  $H(z)$  with  $0.09 \leq z \leq 1.75$  have been obtained by using absolute ages of 32 galaxies taken from the Gemini Deep Deep Survey (GDDS) and the archival data.

For the adopted  $f(R)$  model, we calculate the values of  $\chi^2$  for  $n = 1$  and  $n = 2$ . For this we define

$$\chi^2(H_0, n) = \sum_{i=1}^9 \frac{(H_{\text{exp}}(z_i, n, H_0) - H_{\text{obs}}(z_i))^2}{\sigma_i^2}. \quad (29)$$

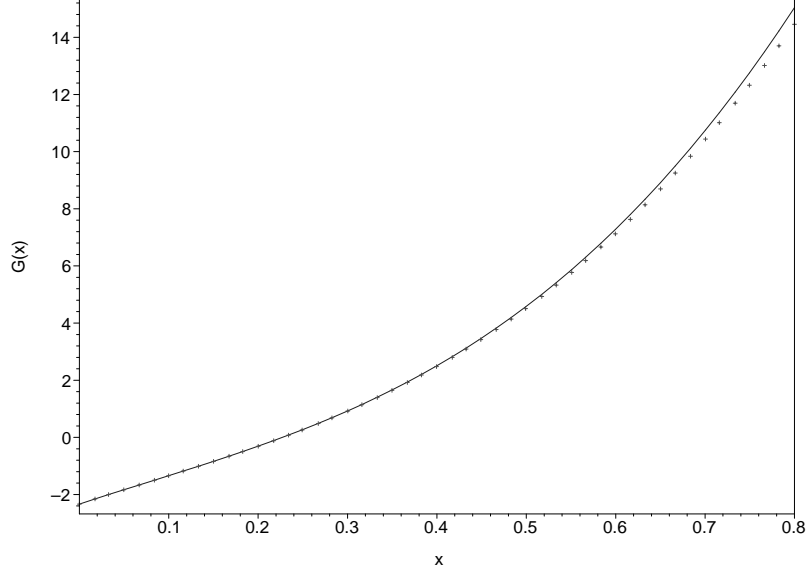


FIG. 8: Plot of  $G(x)$  at  $z = 0, \lambda = 2$  (indicated by solid line) vs truncated expression of  $G(x)$  at  $z = 0, \lambda = 2$  (indicated by dotted line).

Here  $H_0$  is the present day value of the Hubble constant,  $H_{\text{exp}}(z_i, n, H_0)$  is the expected value of the Hubble constant in the  $f(R)$  cosmology at redshift  $z_i$  for a particular  $n$  and  $H_0$ .  $H_{\text{obs}}$  is the observed value and  $\sigma_i$  is the corresponding  $1\sigma$  uncertainty in the measurement. The sum is over all observed data points (nine in number).

As the value of  $\chi^2$  is highly sensitive to the value of  $H_0$ , we marginalize over  $H_0$  to obtain the modified  $\chi^2$ . For this we define the likelihood function as:

$$\mathcal{L} = \int e^{-\chi^2/2} P(H_0) dH_0.$$

Here  $P(H_0)$  is the prior probability function for  $H_0$  which is Gaussian:

$$P(H_0) = \frac{1}{\sqrt{2\pi}\sigma_{H_0}} \exp\left[-\frac{1}{2} \frac{(H_0 - H_0^{\text{obs}})^2}{\sigma_{H_0}^2}\right],$$

with  $H_0^{\text{obs}}$  as the value of  $H_0$  (and  $\sigma_{H_0}$  is the error in it) as suggested by independent observations. In this paper, we also study the effect of different priors on the result. We use two set of priors:

**Set A:**  $H_0^{\text{obs}} = 68 \pm 4$  Km/s/Mpc, as obtained from the median statistics analysis of 461 measurements of  $H_0$  [26].

**Set B:**  $H_0^{\text{obs}} = 77 \pm 4$  Km/s/Mpc, as suggested by the Chandra X - ray Observatory results [27].

We observe that the value of the modified  $\chi_{H(z)}^2$  is sensitive to the choice of the prior.

### Baryon Acoustic Oscillation (BAO)

Before recombination, the universe was in the completely ionized state. The cosmological perturbations in the relativistic primordial baryon-photon plasma produced acoustic oscillations. These oscillations were imprinted in the form of peaks in the late time power spectrum of non-relativistic matter. This acoustic peak which is predicted at

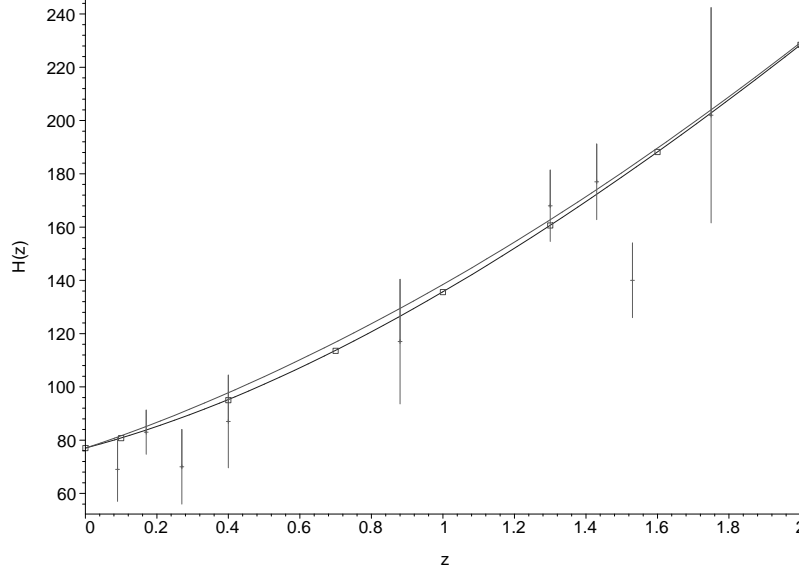


FIG. 9: Variation of  $H(z)$  with  $z$ , for  $H_0 = 77 \text{ Kms}^{-1} \text{ Mpc}^{-1}$ . Upper curve corresponds to  $n=1$  case and the lower curve is for  $n=2$  model. The points (boxes) on the lower curve correspond to  $\Lambda$ CDM.

the measured scale of  $100 h^{-1} \text{ Mpc}$  is detected in the large scale correlation function of 46478 sample of luminous red galaxies in the Sloan Digital Sky Survey[28]. It is described by a dimensionless parameter  $\mathcal{A}$ ,

$$\mathcal{A}(n) \equiv \frac{\Omega_m^{1/2}}{z_*} \left[ z_* \frac{\Gamma^2(z_*, n)}{\mathcal{E}(z_*, n)} \right]^{1/3} \quad (30)$$

where  $z_* = 0.35$ ,  $\Gamma(z_*, n) = \int_0^{z_*} dz / \mathcal{E}(z_*, n)$  is the dimensionless comoving distance to  $z_*$ ,  $\mathcal{E}(z_*, n)$  is given by  $H(z)/H_0$ .

We define  $\chi_{\text{BAO}}^2 = (\mathcal{A}(n) - \mathcal{A}_{\text{obs}})^2 / \sigma_{\mathcal{A}}^2$ , with  $\mathcal{A}_{\text{obs}} = 0.469 \pm 0.017$ . The values of  $\chi_{\text{BAO}}^2$  for  $n = 1$  and  $n = 2$  are listed in Table 1.

| n | $H_0$ prior | $\chi_H^2(z)$ | $\chi_{\text{BAO}}^2$ | $\chi_{\text{Total}}^2$ | $\chi_\nu^2$ |
|---|-------------|---------------|-----------------------|-------------------------|--------------|
| 1 | $68 \pm 4$  | 7.29          | 0.54                  | 7.83                    | 0.87         |
| 1 | $77 \pm 4$  | 6.63          | 0.54                  | 7.17                    | 0.8          |
| 2 | $68 \pm 4$  | 7.79          | 0.88                  | 8.67                    | 0.96         |
| 2 | $77 \pm 4$  | 6.30          | 0.88                  | 7.18                    | 0.8          |

TABLE I: Here  $\chi_{H(z)}^2 = -\frac{1}{2} \ln(\mathcal{L})$ .

## V. RESULTS AND DISCUSSION

The observed late time acceleration of the universe is one of the major unsolved problem in the cosmology. It may hint at the breakdown of Einstein GR. This has lead to modification of the Einstein theory of gravity. One of the

attractive possibility to modify this theory is to replace the Ricci scalar  $R$  with the generic function  $f(R)$  in the Hilbert action.

$f(R)$  theories are usually studied by two methods: *metric* and *Palatini* approach. The Palatini approach is explored extensively in the literature both theoretically and observationally [29] (see the other ref's given in these papers). This formulation gives second order differential field equation which can explain the late time behavior of the universe.

The metric approach of  $f(R)$  theory leads to a fourth order non linear differential equation in terms of scale factor. This equation is difficult to solve both analytically and numerically even for the special cases. Therefore, not much observational tests have been performed on  $f(R)$  theories based on metric formulation. Our work is an attempt to check the concordance of  $f(R)$  theory of gravity using the metric approach with some of the cosmological observations. In particular, we have explored the Starobinsky model in which function  $f(R)$  is analytic, satisfying the condition  $f(0) = 0$ . This model also passes the Solar system and laboratory tests successfully for large values of  $n$ . In this work we study the  $f(R)$  model for  $n = 1$  and  $n = 2$ .

In order to study the cosmological viability of this theory, we investigate this class of model with two observational tests. The first method is based on the Hubble parameter versus redshift data,  $H(z)$ . The Hubble parameter is related to the differential age of the universe through this form

$$H(z) = -\frac{1}{1+z} \frac{dz}{dt}.$$

By estimating the  $dt/dz$ , one can obtain directly the Hubble parameter,  $H(z)$  at different redshifts. The  $H(z)$  data has one major advantage, unlike in the standard candle approach (SNe Ia): the Hubble function is not integrated over. The other important feature of this test is that differential ages are less sensitive to systematic errors as compared to the absolute ages [25]. This observational  $H(z)$  have been used earlier also to constrain various other dark energy models [30].

In this work, we check the compatibility of  $f(R)$  model with  $n = 1$  and  $n = 2$  with the  $H(z)$  data. Since  $H_0$  is a nuisance parameter, we marginalise over  $H_0$ . We further combine the results obtained from  $H(z)$  data set with the BAO data. To perform the joint test, we define the quantity:

$$\chi_{\text{total}}^2 = \chi_{H(z)}^2 + \chi_{\text{BAO}}^2. \quad (31)$$

The results are given in the Table 1. It appears that  $n = 2$  is favored by the observations. This fact is also in agreement with solar and laboratory test. The other important conclusions are as follow:

- The variation of  $H(z)$  with redshift becomes independent of  $n$  after the redshift around  $z = 1.8$ , see Fig. 7. Before this redshift ( $z < 1.8$ ), there is a small difference between the behavior of  $H(z)$  with  $z$  for  $n = 1$  and  $n = 2$  models. This variation is still within the error bars of the data points. At higher redshift (i.e. when  $R \gg R_0$ ) we recover the standard  $\Lambda$ CDM universe for  $\lambda = 2$ , for all the values of  $n$ . Hence the thermal history of the universe is correctly reproduced by this model.
- The expansion history for this model with  $n = 2, \lambda = 2$  matches exactly with the standard cosmological model ( $\Lambda$ CDM) with equation of state parameter  $\omega = -1, \Omega_m = 0.3$  and  $\Omega_\Lambda = 0.7$  (see fig. 7). The variation in parameters  $\Omega_m$  and  $\Omega_\Lambda$  (keeping  $\Omega_{m0} + \Omega_\Lambda = 1$ ) leads to very small changes in values of  $\lambda$ , keeping it close to 2.
- As shown in Table 1, it seems that the present observational data used in this work prefer  $n = 2$  over  $n = 1$ , as indicated by  $\chi^2$  per degree of freedom,  $\chi_\nu^2$ . However,  $n = 1$  cannot be ruled out.
- As stated earlier, the algebraic expressions become cumbersome for larger values of  $n$ . It is quite possible that observational tests discussed in this paper may be compatible with values of  $n$  larger than two which are also consistent with solar and laboratory tests. However, it should be emphasized that for a given value of  $\lambda$ , the minimum of the scalaron potential gets closer to zero for larger values of  $n$  making the model vulnerable to singularity, see Fig. 3.

Due to large error bars in the data sets used, the variation of the expansion history of the universe studied in this paper can easily be accommodated by the observations, thereby making the models close to  $\Lambda$ CDM at the background

level. At present the sample of  $H(z)$  data is too small and the error bars are large. In the future, large amount of precise  $H(z)$  data is expected to become available. This will not only reveal the fine features of the expansion history of the universe but also tightly constrain the cosmological parameters.

There is a further need to explore this model with time based observational tests, like age of the universe and high redshift objects. It is known that the evolution of the age of the universe with the redshift vary from model to model. So it is possible that the model of the universe which are able to reproduce the total age of the universe at  $z = 0$ , may not accommodate the objects at high redshift[31]. Therefore the time based observational test may play the key role in short-listing the viable dark energy model in the near future. It is also expected that the study of matter power perturbations would allow to distinguish this model from the standard cosmology.

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